Week 7
MATH 34B
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18.2 Let $y' = ky(1 - y/10)$, $y'(0) = 2$, and $y(0) = 5$.

(a) What's k?

(b) What's $y(4)$?

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{10}\right)$$

$$\cancel{\int \frac{dy}{y(1-\frac{y}{10})} = \int k\,dt = kt + C}$$

[FOR A MORE DETAILED SOLUTION, CHECK ~~BACK OF~~ THE LAST PAGE]

Gen. solution:

$$y = \frac{10}{Ae^{-kt} + 1}$$

$y(0) = 5$

$\Rightarrow A = 1$

at $t = 0$, have $y' = 2$, $y = 5$

$$\Rightarrow 2 = k \cdot 5\left(1 - \frac{5}{10}\right)$$

$$\Rightarrow 2 = 5k\left(\frac{1}{2}\right)$$

$$\Rightarrow k = \frac{4}{5} \Rightarrow y = \frac{10}{e^{-\frac{4}{5}t} + 1}$$

so plug in 4.

18.5 A spherical snowball is melting. The radius after t hours is $1 - t$ meters.

(a) What fraction of the initial mass of the snowball remains after half an hour?

(b) How quickly is the volume of the snowball changing after half an hour?

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \frac{\pi}{6}$$

$$V(\text{original}) = \frac{4}{3}\pi 1^3 = \frac{4}{3}\pi$$

$$\frac{\frac{\pi}{6}}{\frac{4}{3}\pi} = \frac{1}{8}$$

b) $V = \frac{4}{3}\pi(1-t)^3$

$$\left.\frac{dV}{dt}\right|_{t=\frac{1}{2}} = 4\pi(1-t)^2(-1)\Big|_{t=\frac{1}{2}} = 4\pi(0.5)^2(-1)$$

18.6 How quickly a leaf grows is proportional how big [ie the surface area] the leaf is. If the area of the leaf grows from $2cm^2$ to $3cm^2$ in 8 days, how long will it take for the leaf's area to increase to $5cm^2$?

$A' = kA$

$\Rightarrow A = ce^{kt}$

$A(0) = 2cm$

$\Rightarrow c = 2$

$A = 2e^{kt}$

$3 = 2e^{k \cdot 8}$

$\frac{3}{2} = e^{8k}$

$\ln(3/2) = 8k$

$k = \ln(3/2)/8$

So, solve for t to get answer:

$5 = 2e^{(\ln(\frac{3}{2})/8)t}$

18.7 Species A doubles every 3 hours and initially there are 5 grams. Species B doubles every 4 hours and initially there are 14 grams.

How long until the two species have the same mass?

$A = 5 \cdot 2^{t/3}$

$B = 14 \cdot 2^{t/4}$

set $A = B \Rightarrow 5 \cdot 2^{t/3} = 14 \cdot 2^{t/4}$

$2^{t/3 - t/4} = \frac{2^{t/3}}{2^{t/4}} = \frac{14}{5}$

$\Rightarrow \frac{t}{12}\ln 2 = \ln\left(\frac{14}{5}\right)$

$t = \frac{12\ln\left(\frac{14}{5}\right)}{\ln 2}$

19.7 Assume the amount of pollution entering the world's oceans grows exponentially. At the start of 1900 suppose that the instantaneous rate at which pollution enters the oceans is 10^3 tons per year, and that the amount doubles every 10 years thereafter. Express the rate that pollution enters, in units of tons per year, at a time t years after 1900.

Set up an integral that expresses the total amount of pollution which entered during the period 1900 to 1980.

rate: $\underbrace{10^3}_{\text{initial rate}} 2^{t/10}$ ← rate of doubling.

$$\int_0^{80} 10^3 \cdot 2^{t/10}\, dt$$

19.8 A hot object cools down according to Newton's Law so that t hours after the start, the rate that heat leaves is $76e^{-t/3}$ Joules per hour. How much heat leaves the object during the first 7 hours? [All you need to know is that a Joule is a quantity of heat. In fact, 4.2 Joules is the heat required to raise 1 gram of water 1 degree Celcius]

$$\frac{dH}{dt} = 76\, e^{-t/3}$$

$$\int_1^7 76 e^{-t/3}\, dt$$

$$= \frac{76}{-1/3} e^{-t/3} \Big|_0^7$$

$$= \frac{76\, e^{-7/3}}{(-1/3)} - \frac{76}{(-1/3)} e^{-0}$$

Problem 2, HW 18
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Problem statement: Let $y' = ky(1 - \frac{y}{10})$, $y'(0) = 2$, and $y(0) = 5$.

(a) What's k?

First, we see note that y and y' are both functions in terms of t. In particular, since y satisfies the differential equation listed, we know that for any $y' = ky(1 - \frac{y}{10})$ is true for *any* t. In particular, this is true for $t = 0$. When $t = 0$, we have that $y(0) = 5$, and $y'(0) = 2$. So, we can replace y' with 2 and y with 5 in $y' = ky(1 - \frac{y}{10})$, in which case, we get $2 = k(5)(1 - \frac{5}{10})$, from which, we can solve for k and get $k = \frac{4}{5}$.

(b) What's $y(4)$?

Given $y' = ky(1 - \frac{y}{M})$, we know the general solution is $y = \frac{M}{Ae^{-kt}+1}$ (see slides, and if you're not convinced, the appendix). Here, we have $M = 10$. Since we know that $y(0) = 5$, we see that $5 = \frac{10}{Ae^{-k \cdot 0}+1} = \frac{10}{A+1}$, in which case, we see that $A = 1$. So, we have $y = \frac{10}{e^{-kt}+1}$, where $k = \frac{4}{5}$, from part (a). So, plugging in $t = 4$, we get $y(4) = \frac{10}{e^{-k(4)}+1}$, and substituting $k = 4/5$, this is simply $y(4) = \frac{10}{e^{-\frac{16}{5}}+1}$ (be careful how you plug this in!).

Given $\frac{dy}{dt} = ky(1 - \frac{y}{M})$, we see that we can separate this into $\frac{1}{y(1-\frac{y}{M})}dy = kdt$. We can then integrate these into $\int \frac{1}{y(1-\frac{y}{M})}dy = \int kdt = kt + C$.

So, let us find $\int \frac{1}{y(1-\frac{y}{M})}dy$. If only we can decompose $\frac{1}{y(1-\frac{y}{M})}$ into $\frac{A}{y} + \frac{B}{1-\frac{y}{M}}$... Oh but we can! If we let $\frac{1}{y(1-\frac{y}{M})} = \frac{A}{y} + \frac{B}{1-\frac{y}{M}}$, we can multiply by $y(1 - \frac{y}{M})$ on both sides and get $1 = A(1 - \frac{y}{M}) + By = A - \frac{Ay}{M} + By$, Since the constant term on the left hand side is 1, we see that the constant term on the right hand must be too. The constant term on the right hand side is A (all other terms have y's attached), in which case, $A = 1$. Likewise, since there are no y terms on the right hand side (indeed, it's just 1), we must have that $-\frac{Ay}{M} + By = \left(B - \frac{A}{M}\right) y = 0$, so $\left(B - \frac{A}{M}\right) = 0$. Since $A = 1$, this means $B - \frac{1}{M} = 0$, so $B = \frac{1}{M}$.

Now, we have that $\int y(1-\frac{y}{M})dy = \int \frac{1}{y} + \frac{1/M}{1-\frac{y}{M}}dy = \int \frac{1}{y} + \frac{1}{M-y}dy$ (the last inequality comes from multiplying M on top and bottom on the thing on the right hand side). This is simply $\ln(y) - \ln(M - y)$, which by log rules, is $\ln(\frac{y}{M-y})$. So, we have that $\ln(\frac{y}{M-y}) = kt + C$, so taking both sides to e, we have $\frac{y}{M-y} = e^{kt+C} = e^C e^{kt}$. Let $A = e^C$, so we have $\frac{y}{M-y} = Ae^{kt}$. Now, solving for y, we have $y = \frac{M}{1+Ae^{-kt}}$, and we are done.